Matching with Contracts, Substitutes and Two-Unit Demand

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July 6, 2016

Abstract

In the context of many-to-one matching with contracts, we show that for any choice function that satisfies the irrelevance of rejected contracts condition (Aygün and Sönmez, 2013) and selects at most two contracts from any given set of contracts (two-unit demand), bilateral substitutability and weak substitutability are equivalent. As a corollary, we obtain a new maximal domain for the existence of stable matchings in the unit-capacity couples model. Finally, we show with an example that the equivalence between bilateral and weak substitutability crucially depends on the two-unit demand condition.

Keywords: Matching; Contracts; Stability, Couples; Weak Substitutability; Bilateral Substitutability.

JEL classification: C78; D47.

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1 Introduction

In this note we study conditions for the existence of stable matchings in the many-to-one matching with contracts model of Hatfield and Milgrom (2005). Stability is a central concept in the matching literature. Theoretically, stable matching are immune to rematching. Moreover, there is empirical evidence that in centralized labor markets, clearinghouses are most often successful if they produce stable matchings.¹ Unfortunately, if contracts are complements from the point of view of firms,² stable matchings may not exist.

The existence of stable matchings can be guaranteed by imposing conditions on firms’ choice functions. Under the assumption that each firm’s choice function satisfies a mild consistency condition called irrelevance of rejected contracts (IRC) (Aygün and Sönmez, 2013), bilateral substitutability is a sufficient condition for the existence of stable matchings (Hatfield and Kojima (2010), Theorem 1 and Aygün and Sönmez (2012), Theorem 1). A weaker condition, weak substitutability, is necessary to guarantee the existence of stable matchings for all possible “unit-demand” choice functions (preferences) of other agents (Hatfield and Kojima (2008), Proposition 1).³ That is, if a firm’s choice function does not satisfy weak substitutability, then there are “unit-demand” choice functions for other firms and preferences for workers such that no stable matching exists.

Our main result states that if a choice function satisfies the IRC condition and it selects at most two contracts from any given set of contracts (two-unit demand), then weak substitutability implies bilateral substitutability. Hence, for any choice function that satisfies the IRC condition and two-unit demand, weak substitutability and bilateral substitutability are equivalent. An implication of this equivalence is that bilateral substitutability is a maximal domain for the existence of stable matchings provided that each firm’s choice function satisfies the IRC condition and two-unit demand.⁴

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¹See Roth (2002) for a comparison between real-life mechanisms that produce stable matchings and real-life mechanisms that produce unstable matchings.

²We refer to agents on the “one” side of the market as firms and to agents on the “many” side of the market as workers, although the applications of the many-to-one matching with contracts model are not restricted to labor markets. For example, the model has been applied to the analysis of school choice with soft-caps by Hafalir, Yenmez and Yildirim (2013), cadet-branch matching by Sönmez and Switzer (2013) and matching under distributional constraints by Kamada and Kojima (2015).

³A firm’s choice function satisfies unit-demand if it selects at most one contracts from any given set of contracts. This can be the case, for example, when a firm has a single position. Workers have unit-demand as each of them works for at most one firm. Unit-demand implies bilateral substitutability.

⁴In the matching literature, a domain $\mathcal{D}$ of individual agents’ choice functions (or preferences) is called a maximal domain if for every set of choices $\mathcal{C}$, there exists a stable matching that satisfies the choice functions in $\mathcal{C}$.
We elaborate on the implications of our result for matching markets with couples. A prototypical example of a matching market with couples is the market for medical residency positions in the U.S. In this market, each doctor can apply to hospitals as single or as part of a couple, and each hospital can have multiple positions (see Roth (2008) for more details). Hatfield and Kojima (2010) observe that the market for medical residency positions is an instance of a many-to-many matching with contracts model where there are two contracts between each couple and each hospital, one for each member of the couple. We refer to this model of the market for medical residency positions as the couples model.5

In the couples model, and more generally in many-to-many matching, several stability concepts have been proposed depending on what types of blocking coalitions are allowed. Furthermore, there are no obvious reasons to focus on one stability concept over another and all of them reduce to the same stability concept when hospitals have a single position each (Kojima, Pathak, and Roth (2013) discuss this issue in the context of the couples model). The couples model allows each hospital to have complex preferences over subsets of doctors. However, in applications hospitals often have preferences with a simple structure: the rank of a doctor at a given hospital is independent of her colleagues. In this case, no generality is lost by treating each hospital with multiple positions as multiple hospitals with a single position each, although this approach would lead to a particular stability concept (Kojima, Pathak and Roth (2013), footnote 22). We call this model of the market for medical residency positions the unit-capacity couples model.

In many-to-many matching with contracts (and therefore in the couples model), the substitutability of each agent’s choice function is a necessary and sufficient condition, i.e., a maximal domain, for the existence of matchings that satisfy a certain stability concept (Hatfield and Kominers, 2016). However, in the unit-capacity couples model substitutability is not necessary (Hatfield and Kojima, 2008). An implication of our equivalence result is that in the unit-capacity couples model, bilateral substitutability is both necessary and sufficient for the existence of stable matchings.

The identification of bilateral substitutability as a new maximal domain is im-

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5In the couples model, each couple signs at most one contract for each of its members. Therefore, each couple’s choice function satisfies two-unit demand.
portant because until now the only maximal domain known for the unit-capacity couples model was the domain of weakly responsive preferences under a restricted unemployment aversion condition (Klaus and Klijn, 2005 and Klaus, Klijn and Nakamura, 2009). Our result improves upon Klaus and Klijn’s (2005) and Klaus, Klijn and Nakamura’s (2009) maximal domain result in two ways. First, it does not require the restricted unemployment aversion condition, which may be implausible in several situations. For example, it is violated when a couple prefers a high paying job for one of its members and the unemployment for the other better than two geographically distant jobs. Second, bilateral substitutability is a strictly larger domain than weak responsiveness (Hatfield and Kojima (2010), Theorem 2 and Example 4).

Finally, we note that the applicability of our results is not restricted to matching with couples markets. Another application is, for example, the higher education scheme in Hungary, where students typically apply for pairs of MSc studies and hence act like couples in the market for medical residency positions (see Biró (2008) for more details).

2 Notation

To present our results, we only need the following partial description of the matching with contracts model of Hatfield and Milgrom (2005).

There is a single firm $f$ and there are (finite and disjoint) sets $W$ of workers and $X$ of contracts. Each contract $x \in X$ is associated with $f$ and with a worker $x_W \in W$. Let $Y \subseteq X$, we define $Y_W \equiv \bigcup_{y \in Y} \{y_W\}$ to be the set of workers with contracts in $Y$.

Given a set of contracts $Y \subseteq X$, $f$’s choice set $C(Y)$ is a subset of $Y$, i.e., $C(Y) \subseteq Y$. We assume that $f$ can sign only one contract with any given worker, i.e.,

$$\forall Y \subseteq X, \ \forall x, x' \in C(Y), \ \ x \neq x' \implies x_W \neq x'_W.$$ Let $Y \subseteq X$, we define $f$’s rejected set of contracts as $R(Y) \equiv Y \setminus C(Y)$. We refer to the function that maps each set of contracts to the choice (rejected) set as the choice (rejection) function.

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6 A couple’s preferences are weakly responsive if there exist individual preferences for each of its members such that an improvement in one couple member’s job (according to that member’s individual preferences) is an improvement for the couple as well. A couple’s preferences satisfy restricted unemployment aversion if the couple is worse off when one of his members loses a position that is “acceptable” according to that member’s individual preferences.

7 Weak responsiveness (and hence bilateral substitutability) does not ensure that the set of stable matchings forms a lattice nor that there is a stable and strategy proof mechanism (Klaus and Klijn, 2005).
3 Conditions on choice functions

A choice function satisfies the irrelevance of rejected contracts condition (IRC) (Aygün and Sönmez, 2013) if the removal of rejected contracts does not affect the choice set. The IRC condition is a mild consistency requirement. In particular, it is easy to check that any choice function generated by the maximization of a strict preference relation satisfies the IRC condition.

Irrelevance of rejected contracts (IRC):

\[ \forall X', X'' \subseteq X, \quad C(X'') \subseteq X' \subseteq X'' \implies C(X') = C(X''). \]

A choice function satisfies two-unit demand if it never selects sets of size larger than two.

Two-unit demand (TUD): \( \forall Y \subseteq X, \quad |C(Y)| \leq 2. \)

A choice function satisfies bilateral substitutability (Hatfield and Kojima, 2010) if whenever a contract \( z \) is rejected when all available contracts involve different workers, contract \( z \) is still rejected when contracts with new workers are added to the choice set.

Bilateral substitutability (BS): there are no set of contracts \( Y \subseteq X \) and a pair of contracts \( x, z \in X \) such that

\[ x_W, z_W \notin Y_W, \quad z \notin C(Y \cup \{z\}) \text{ and } z \in C(Y \cup \{x, z\}). \]

A choice function satisfies weak substitutability (Hatfield and Kojima, 2008) if a contract is chosen from some set of available contracts where no two different contracts involve the same worker, then that contract is still chosen from any smaller set that includes it.

Weak substitutability (WS): for each \( X' \subseteq X'' \subseteq X \) such that

\[ [x, z \in X'' \text{ and } x_W = z_W] \implies x = z, \quad R(X') \subseteq R(X''). \]

Note that bilateral substitutability implies weak substitutability.\(^8\)

\(^8\)Further, two-unit demand and bilateral/weak substitutability do not imply Hatfield and Kojima’s (2010) unilateral substitutability.
Proposition 1. Suppose that a choice function satisfies the IRC condition and TUD. Then, if the choice function satisfies WS, it also satisfies BS.

Proof: Suppose that a choice function \( C \) satisfies the IRC condition and TUD, but it does not satisfy BS. Then, there are a set of contracts \( Y \subseteq X \) and contracts \( x, z \in X \) such that \( x_W, z_W \notin Y_W \) and

\[
z \notin C(Y \cup \{z\}) \quad \text{and} \quad z \in C(Y \cup \{x, z\}). \tag{1}
\]

Let \( X' \equiv C(Y \cup \{z\}) \cup \{z\} \) and \( X'' \equiv C(Y \cup \{z\}) \cup \{x, z\} = X' \cup \{x\} \).

**Step 1:** \( z \notin C(X') \).

By definition of \( X' \), \( C(Y \cup \{z\}) \subseteq X' \subseteq Y \cup \{z\} \). By IRC, \( C(X') = C(Y \cup \{z\}) \). Since \( z \notin C(Y \cup \{z\}) \), we conclude \( z \notin C(X') \).

**Step 2:** \( C(Y \cup \{x, z\}) = \{x, z\} \).

By (1), \( z \in C(Y \cup \{x, z\}) \). Assume \( x \notin C(Y \cup \{x, z\}) \). Then, \( C(Y \cup \{x, z\}) \subseteq Y \cup \{z\} \subseteq Y \cup \{x, z\} \). By IRC, \( C(Y \cup \{z\}) = C(Y \cup \{x, z\}) \). However, this is a contradiction to (1). Therefore, \( x \in C(Y \cup \{x, z\}) \). So, \( C(Y \cup \{x, z\}) \supseteq \{x, z\} \). By TUD, \( C(Y \cup \{x, z\}) = \{x, z\} \).

**Step 3:** \( z \in C(X'') \).

From Step 2 and the definition of \( X'' \), it follows that \( C(Y \cup \{x, z\}) = \{x, z\} \subseteq X'' \subseteq Y \cup \{x, z\} \). By IRC,

\[
C(X'') = C(Y \cup \{x, z\}) = \{x, z\}.
\]

From Steps 1 and 3, it immediately follows that \( R(X') \not\subseteq R(X'') \). \( \tag{2} \)

**Step 4:** \( X'' \) does not involve two different contracts with the same worker.

Since \( z \notin C(Y \cup \{z\}) \) and \( x_W, z_W \notin Y_W \), we have \( x_W, z_W \notin \left[ C(Y \cup \{z\}) \right]_W \). Moreover, since no firm signs more than one contract with the same worker, \( C(Y \cup \{x, z\}) = \{x, z\} \) implies that \( x_W \neq z_W \).

From (2) and Step 4, we conclude that WS does not hold. \( \blacksquare \)

As an immediate corollary to Proposition 1, we obtain the following equivalence result.
Corollary 1. For any choice function that satisfies the IRC condition and TUD, WS is equivalent to BS.

We give an example of a strict preference relation that induces a choice function that satisfies the IRC condition and WS but does not satisfy neither TUD nor BS. This implies that TUD is crucial for the equivalence between WS and BS.

Example 1. (Necessity of TUD for the equivalence between WS and BS.)

There is a single firm \( f \) and there are three workers denoted by \( a, b, c \). There is a single contract associated with worker \( a \) denoted (with a slight abuse of notation) by \( a \), there is a single contract associated with worker \( b \) denoted (with a slight abuse of notation) by \( b \), and there are two contracts associated with worker \( c \), denoted by \( c_1 \) and \( c_2 \). In Table 1, the first column depicts the strict preference relation, \( P \), over subsets of \( \{ a, b, c_1, c_2 \} \). Higher placed sets are more preferred. For example, it indicates that the set \( \{ c_2 \} \) is more preferred than the set \( \{ a, b \} \). Vertical dots indicate that all other subsets of contracts are not acceptable, i.e., less preferred than the empty set. The second column depicts all subsets of \( \{ a, b, c_1, c_2 \} \). Let \( C \) be the choice function generated by the maximization of \( P \). The third column depicts for each \( Y \subseteq \{ a, b, c_1, c_2 \} \), the choice set \( C(Y) \).

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Since $C$ is generated by the maximization of a strict preference relation, it satisfies the IRC condition. We show that $C$ satisfies WS but does not satisfy neither TUD nor BS.

Since $|C(\{a, b, c_1, c_2\})| > 2$, it violates TUD. Next, we show a violation of BS. Let $Y = \{c_1, c_2\}$ and note that

$$a \notin \{c_2\} = C(Y \cup \{a\}) \text{ and } a \in \{a, b, c_1\} = C(Y \cup \{a, b\}).$$

Moreover, workers $a$ and $b$ have no contracts in $Y$ as $Y_W = \{c\}$. Therefore, $C$ does not satisfy BS.

Finally, we show that WS is satisfied. Let $Y \subseteq \{a, b, c_1, c_2\}$. There are two cases.

**Case 1.** $Y \neq \{a, b, c_1, c_2\}$ and $c_2 \notin Y$.

In this case $C(Y) = Y$. Therefore, $R(Y) = \emptyset$ which implies that $R(Y) \subseteq R(Y')$ for each $Y'$ with $Y \subseteq Y'$.

**Case 2.** $Y \neq \{a, b, c_1, c_2\}$ and $c_2 \in Y$.

In this case $C(Y) = \{c_2\}$. Hence, for each $Y'$ with $Y \subseteq Y'$, $R(Y) \subseteq R(Y')$ or $Y' = \{a, b, c_1, c_2\}$. Since $\{a, b, c_1, c_2\}$ contains two contracts with the same worker $c$, it is not possible to construct a violation of WS.

Therefore, we conclude that $C$ satisfies WS.

## 5 Final remarks

Couples are present in many matching markets. This note adds to the understanding of when stable matchings exist in their presence. We show that two known conditions in the matching with contracts literature, weak substitutability and bilateral substitutability, coincide in any matching problem where agents on the “one” side of the market can sign at most two contracts. Hence, bilateral substitutability is a maximal domain for the existence of stable matchings in the unit-capacity couples model. Unfortunately, bilateral substitutability is a restrictive condition as it rules out most complementarities. This suggests difficulties for the operation of real life matching markets with couples. However, some recent work has shown that for every instance of a matching with couples problem there is always a “nearby” instance for which a stable matching exists (Nguyen and Vohra, 2014). Moreover, in several types of matching markets with complementarities and a large number of participants, stable matchings exist under broad conditions (see, for example,
Kojima, Pathak and Roth, 2013; Ashlagi, Braverman and Hassidim, 2014; Azevedo and Hatfield, 2015; and Che, Kim, and Kojima, 2015).

Acknowledgments

I am grateful to Flip Klijn for his generous feedback on previous versions of this note. I thank an anonymous referee for helpful comments and suggestions. Financial support from the Consejo Nacional de Ciencia y Tecnología (CONACyT), AGAUR-Generalitat de Catalunya (Project 2014 SGR 1064), Universitat Autònoma de Barcelona through PIF grant 412-01-9/2010 and the Spanish Ministry of Economy and Competitiveness through FPI grant BES-2012-055341 (Project ECO2011-29847-C02) is gratefully acknowledged.

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